NEODREĐENI INTEGRALI

Na osnovu tabličnih

1

$$\int\left(3\sqrt{x}+\frac{2}{x}\right)^2 dx = \int\left(9x+\frac{12}{\sqrt{x}}+\frac{4}{x^2}\right)dx =$$

$$= 9\int x\,dx + 12\int\frac{dx}{\sqrt{x}} + 4\int\frac{dx}{x^2} = \frac{9}{2}x^2 + 24\sqrt{x} - \frac{4}{x} + C$$

2

$$\int\cos^2(2x-1)\,dx = \left[\begin{array}{l} \sin^2x+\cos^2x=1 \\ \cos 2x = \cos^2x - \sin^2x \end{array}\right\} \overset{①}{+} \Big\} \overset{②}{-}$$

$$\Rightarrow^{①} \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\Rightarrow^{②} \sin^2 x = \frac{1-\cos 2x}{2} \Bigg] =$$

$$= \int\frac{1+\cos(4x-2)}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int\cos(4x-2)\,dx =$$

$$= \frac{1}{2}x + \frac{1}{8}\sin(4x-2) + C$$

3

$$\int(1-x)(1-2x)(1-3x)\,dx = \int(1-6x+11x^2-6x^3)\,dx =$$

$$= x - 6\frac{x^2}{2} + 11\frac{x^3}{3} - 6\frac{x^4}{4} + C$$

4

$$\int\frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx = \int\left(x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}}\right)dx =$$

$$= \frac{4}{5}x^{\frac{5}{4}} - 2\cdot\frac{12}{17}x^{\frac{17}{12}} + \frac{4}{3}\cdot x^{\frac{3}{4}} + C$$

5

$$\int\frac{x^2}{1+x^2}dx = \int\frac{x^2+1-1}{1+x^2}dx = \int dx - \int\frac{dx}{1+x^2} =$$

$$= x - \operatorname{arctg} x + C$$

6

$$\int\frac{x^3+x-2}{x^2+1}dx = \left[\begin{array}{l} (x^3+x-2):(x^2+1) = x \\ \underline{-x^3-x} \\ \quad -2 \end{array}\right] =$$

$$= \int\frac{x(x^2+1)-2}{x^2+1}dx = \int x\,dx - 2\int\frac{dx}{x^2+1} = \frac{x^2}{2} - 2\operatorname{arctg}x$$

(7) $$\int \frac{2^{x+1}-5^{x-1}}{10^x}\,dx = \int 2\left(\frac{2}{10}\right)^x dx - \int \frac{1}{5}\cdot\left(\frac{5}{10}\right)^x dx =$$

$$= 2\int\left(\frac{1}{5}\right)^x dx - \frac{1}{5}\int\left(\frac{1}{2}\right)^x dx = 2\,\frac{\left(\frac{1}{5}\right)^x}{\ln\frac{1}{5}} - \frac{1}{5}\,\frac{\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}} + C =$$

$$= 2\,\frac{\left(\frac{1}{5}\right)^x}{\ln 5^{-1}} - \frac{1}{5}\,\frac{\left(\frac{1}{2}\right)^x}{\ln 2^{-1}} + C = \frac{1}{5}\,\frac{\left(\frac{1}{2}\right)^x}{\ln 2} - 2\,\frac{\left(\frac{1}{5}\right)^x}{\ln 5} + C$$

(8) $$\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int\frac{dx}{\cos^2 x} - \int dx =$$

$$= \operatorname{tg} x - x + C$$

Smjena

(9) $$\int \cos^5 x\,(dx)\,\sin x = \left[\begin{array}{l} t=\cos x \\ -\sin x\,dx = dt \end{array}\right] = -\int t^5\,dt = -\frac{t^6}{6} + C =$$

$$= -\frac{\cos^6 x}{6} + C$$

(10) $$\int \frac{\ln^2 x}{x}\,dx = \left[\begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \end{array}\right] = \int t^2\,dt = \frac{t^3}{3} + C = \frac{\ln^3 x}{3} + C$$

(11) $$\int \frac{x\,dx}{x^2+a} = \left[\begin{array}{l} x^2+a=t \\ 2x\,dx = dt \end{array}\right] = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln|x^2+a| + C$$

(12) $$\int\sqrt{a^2-x^2}\,dx = \left[\begin{array}{l} x = a\sin t \quad t=\arcsin\frac{x}{a} \\ dx = a\cos t\,dt \end{array}\right] = a\int\sqrt{1-\sin^2 t}\;a\cdot\cos t\,dt =$$

$$= a^2\int\cos^2 t\,dt = a^2\int\frac{1+\cos 2t}{2}\,dt = \frac{a^2}{2}t + \frac{a^2}{2}\sin t\cos t + C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\,\frac{x}{a}\cdot\sqrt{1-\frac{x^2}{a^2}}$$

(13) $\int \operatorname{ctg} x \, dx = \int \frac{\cos x}{\sin x} dx = \left[\begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right] = \int \frac{dt}{t} = \ln|\sin x| + C$

(14) $\int \frac{dx}{\sin x} = \int \frac{dx}{\sin(2\frac{x}{2})} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{dx}{2 \operatorname{tg}\frac{x}{2} \cos^2\frac{x}{2}} =$

$= \left[\begin{array}{l} \operatorname{tg}\frac{x}{2} = t \\ \frac{dx}{2\cos^2\frac{x}{2}} = dt \end{array} \right] = \int \frac{dt}{t} = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$

(15) $\int \frac{x+5}{\sqrt[5]{1-2x}} dx = \left[\begin{array}{l} 1-2x = t \\ -2\,dx = dt \\ x = \frac{1-t}{2} \end{array} \right] = -\frac{1}{2}\int \frac{\frac{1-t}{2} + 5}{\sqrt[5]{t}} dt =$

$= -\frac{1}{2} \int \frac{11-t}{2\sqrt[5]{t}} dt = -\frac{11}{4}\int t^{-\frac{1}{5}} dt + \frac{1}{4}\int t^{\frac{4}{5}} dt =$

$= -\frac{55}{16} t^{\frac{4}{5}} + \frac{5}{36} t^{\frac{9}{5}} + C = -\frac{55}{16} \cdot \sqrt[5]{(1-2x)^4} + \frac{5}{36} \sqrt[5]{(1-2x)^9} +$

(16) $\int \frac{dx}{2+3x^2} = \int \frac{dx}{2(1+\frac{3}{2}x^2)} = \frac{1}{2}\int \frac{dx}{1+\frac{3}{2}x^2} = \left[\begin{array}{l} \sqrt{\frac{3}{2}}\,x = t \\ \sqrt{\frac{3}{2}}\,dx = dt \end{array} \right] =$

$= \cancel{\frac{\sqrt{3}}{\sqrt{2}}} \frac{1}{2} = \sqrt{\frac{2}{3}} \cdot \frac{1}{2} \int \frac{dt}{1+t^2} = \sqrt{\frac{2}{3}} \frac{1}{2} \operatorname{arctg} t + C =$

$= \frac{1}{\sqrt{6}} \operatorname{arctg} \sqrt{\frac{3}{2}}\, x + C$

(17) $\int \frac{x^3}{x+3} dx = \int \frac{x^3+27-27}{x+3} dx = \int \frac{\cancel{(x+3)}(x^2-3x+9)}{\cancel{x+3}} dx =$

$= \int x^2 dx - 3\int x\,dx + 9\int dx = \frac{x^3}{3} - \frac{3}{2}x^2 + 9x + C$

(18) $\int \frac{dx}{\sqrt{3-5x^2}} = \int \frac{dx}{\sqrt{3}\cdot\sqrt{1-\frac{5}{3}x^2}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{1-\frac{5}{3}x^2}} = \left[\begin{array}{l} \sqrt{\frac{5}{3}}\,x = t \\ dx = \sqrt{\frac{3}{5}}\,dt \end{array} \right] =$

$= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{5}} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\sqrt{5}} \arcsin \sqrt{\frac{5}{3}}\, x + C$

(19) $\int \frac{dx}{1+\sin x} = \int \frac{dx}{1+\cos\left(\frac{\pi}{2}-x\right)} = \left[\begin{array}{l} \frac{\pi}{2}-x=t \\ -dx=dt \end{array} \right] = -\int \frac{dt}{1+\cos t} =$

$= -\int \frac{dt}{\cos^2\frac{t}{2}+\sin^2\frac{t}{2}+\cos^2\frac{t}{2}-\sin^2\frac{t}{2}} = -\frac{1}{2}\int \frac{dt}{\cos^2 t} = -\frac{1}{2}\operatorname{tg}\frac{\left(\frac{\pi}{2}-x\right)}{2}+C$

(20) $\int \frac{x+3}{x^2+6x-5}\,dx = \left[\begin{array}{l} \cancel{(x^2+6x-5)\cdot(x+3)} \\ x^2+6x-5=t\,/\,d \\ (2x+6)\,dx=dt \end{array} \quad 2(x+3)\,dx=dt \right] =$

$= \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln\left|x^2+6x-5\right| + C$

(21) $\int \frac{\ln^3 x}{x}\,dx = \left[\begin{array}{l} \ln x=t \\ \frac{1}{x}dx=dt \end{array} \right] = \int t^3\cdot dt = \frac{t^4}{4} = \frac{\ln^4 x}{4}+C$

(22) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx = \left[\begin{array}{l} \sin x=t \\ \cos x\,dx=dt \end{array} \right] = \int \frac{dt}{t^{\frac{2}{3}}} = 3\sqrt[3]{\sin x}+C$

(23) $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}\,dx = \left[\begin{array}{l} \ln\left(x+\sqrt{1+x^2}\right)=t \\ \frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}}\cdot\cancel{2}x\right)dx=dt \\ \frac{1}{x+\cancel{\sqrt{1+x^2}}}\cdot\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\,dx=dt \end{array} \right] =$

$= \int \sqrt{t}\,dt = \frac{2}{3}\sqrt[3]{\ln^2\left(x+\sqrt{1+x^2}\right)}+C$

(24) $\int \frac{e^{\arcsin x}+x+1}{\sqrt{1-x^2}}\,dx = \int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$

$\arcsin x = t$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

$1-x^2=z$

$-2x\,dx=dz$

14

JORDI LABANDA

$$= \int e^t\,dt - \frac{1}{2}\int z^{-\frac{1}{2}}\,dz + \int \frac{dx}{\sqrt{1-x^2}} =$$

$$= e^{\arcsin x} - \frac{1}{2}\cdot 2\sqrt{1-x^2} + \arcsin x + C$$

(25) $$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{dx}{(\sin x\cos x)^2} = \int \frac{dx}{(2\sin x\cos x)^2}\cdot 4 =$$

$$= 4\int \frac{dx}{(\sin 2x)^2} = 4\cdot\frac{1}{2}\cdot\int \frac{d(2x)}{\sin^2 2x} = -2\operatorname{ctg} 2x + C$$

Parcijalna integracija

(26) $$\int x\sin x\,dx = \left[\begin{array}{l} U = x \Rightarrow dU = dx \\ dV = \sin x\,dx \Rightarrow V = \int \sin x\,dx = -\cos x \end{array}\right] =$$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

(27.) $$\int \ln x\,dx = \left[\begin{array}{l} U = \ln x \Rightarrow dU = \frac{1}{x}dx \\ dV = dx \Rightarrow V = \int dx = x \end{array}\right] = x\ln x - \int \frac{1}{x}\cdot x\,dx =$$

$$= x\ln x - x + C = x(\ln x - 1) + C$$

(28.) $$\int x^2\sin 2x\,dx = \left[\begin{array}{l} U = x^2 \Rightarrow dU = 2x\,dx \\ dV = \sin 2x\,dx \Rightarrow V = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x \end{array}\right] =$$

$$= -\frac{x^2}{2}\cos 2x + \frac{1}{2}\int \cos 2x\cdot 2x\,dx =$$

$$= \frac{x^2}{2}\cos 2x + \int x\cos 2x\,dx = \left[\begin{array}{l} U = x \Rightarrow dU = dx \\ dV = \int \cos 2x\,dx = \frac{1}{2}\sin 2x \end{array}\right] =$$

$$= \frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx =$$

$$= \frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

(29) $\int \sqrt{x}\, \ln^2 x \, dx = \Big[\cancel{U = \sqrt{x} \Rightarrow dU = \frac{2}{3}\sqrt{x^3}\, dx}$

$U = \ln^2 x \Rightarrow dU = 2\ln x \, \frac{1}{x}\, dx$

$dV = \sqrt{x}\, dx \Rightarrow V = \int \sqrt{x}\, dx = \frac{2}{3}\sqrt{x^3} \Big] =$

$= \cancel{\frac{4}{3}\sqrt{x}} = \frac{2}{3}\sqrt{x^3}\ln^2 x - \frac{4}{3}\int \sqrt{x}\,\ln x\, dx = \Big[\ln x = t \Rightarrow \frac{1}{x}dx = dt$

$V = \int \sqrt{x}\, dx = \frac{2}{3}\sqrt{x^3}\, dx \Big] =$

$= \frac{2}{3}\sqrt{x^3}\ln^2 x - \frac{4}{3}\left(\frac{2}{3}\sqrt{x^3}\ln x - \frac{2}{3}\int \sqrt{x}\, dx \right) =$

$= \frac{2}{3}\sqrt{x^3}\ln^2 x - \frac{8}{9}\sqrt{x^3}\ln x + \frac{8}{9}\cdot\frac{2}{3}\sqrt{x^3} + C$

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(30) $\int \frac{x\cos x}{\sin^3 x}\, dx = \Big[U = x \Rightarrow dU = dx$

$V = \int \frac{\cos x}{\sin^3 x}\, dx = -\frac{1}{2}\,\frac{1}{\sin^2 x} \Big] =$

$= -\frac{1}{2}\frac{x}{\sin^2 x} + \frac{1}{2}\int \frac{dx}{\sin^2 x} = -\frac{1}{2}\frac{x}{\sin^2 x} - \frac{1}{2}\operatorname{ctg} x + C$

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(31) $\int x\, e^x\, dx = \Big[U = x \Rightarrow dU = dx$

$V = \int e^x dx = e^x \Big] = x\cdot e^x - \int e^x dx = xe^x - e^x + C$

(31.) $\int \operatorname{arctg} x\, dx = \Big[U = \operatorname{arctg} x \Rightarrow dU = \frac{1}{1+x^2}\, dx$

$dV = dx \Rightarrow V = \int dx = x \Big] =$

$= x\operatorname{arctg} x - \int \frac{x}{1+x^2}\, dx = x\operatorname{arctg} x - \frac{1}{2}\ln|1+x^2| + C$

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(32) $\int \ln^2 x\, dx = \Big[U = \ln^2 x \Rightarrow dU = 2\ln x\,\frac{1}{x}\, dx$

$dV = dx \Rightarrow V = \int dx = x \Big]$

$= x\ln^2 x - \int 2\ln x\, dx = \Big[U = \ln x \Rightarrow dU = \frac{1}{x}dx$

$V = \int dx = x \Big] =$

$= x\ln^2 x - 2x\ln x + 2\int dx = x\ln^2 x - 2x\ln x + 2x + C$

JORDI LABAÑDA

(33.) $I = \int e^{ax} \sin bx \, dx = \left[\begin{array}{l} U = e^{ax} \Rightarrow dU = a e^{ax} dx \\ V = \int \sin bx \, dx = -\frac{1}{b} \cos bx \end{array} \right] =$

$= -\frac{e^{ax}}{b} \cos bx + \frac{a}{b} \int e^{ax} \cos bx \, dx = \left[\begin{array}{l} U = e^{ax} \Rightarrow dU = a e^{ax} \\ V = \frac{1}{b} \sin bx \end{array} \right] =$

$= -\frac{e^{ax}}{b} \cos bx + \frac{a}{b} \left(\frac{e^{ax}}{b} \sin bx - \frac{a}{b} \int e^{ax} \sin bx \, dx \right) =$

$I = -\frac{e^{ax}}{b} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I$

$\left(1 + \frac{a^2}{b^2}\right) I = \frac{a}{b^2} e^{ax} \sin bx - \frac{e^{ax}}{b} \cos bx$

$I = \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax}$

(34.) $I = \int \frac{dx}{(x^2+1)^2}$

$I_1 = \int \frac{dx}{x^2+1} = \left[\begin{array}{l} U = \frac{1}{x^2+1} \Rightarrow dU = -\frac{1}{(x^2+1)^2} 2x \, dx \\ V = \int dx = x \end{array} \right] =$

$= \frac{x}{x^2+1} + 2 \int \frac{x^2}{(x^2+1)^2} dx = \frac{x}{x^2+1} + 2 \int \frac{x^2+1-1}{(x^2+1)^2} dx =$

$I_1 = \frac{x}{x^2+1} + 2 \int \frac{dx}{x^2+1} - 2 \int \frac{dx}{(x^2+1)^2}$

$2I = \frac{x}{x^2+1} + \int \frac{dx}{x^2+1} \rightarrow I = \frac{1}{2} \left(\frac{x}{x^2+1} + \operatorname{arctg} x \right) + C$

(35.) $\int \frac{\arccos x}{\sqrt{(1-x^2)^3}} dx = \int \frac{\arccos x}{(1-x^2)\sqrt{1-x^2}} dx = \left[\begin{array}{l} \arccos x = t \\ -\frac{dx}{\sqrt{1-x^2}} = dt \\ x = \cos t \end{array} \right]$

$= -\int \frac{t \, dt}{1 - \cos^2 t} = -\int \frac{t \, dt}{\sin^2 t} = \left[\begin{array}{l} U = t \Rightarrow dU = dt \\ V = \int \frac{dt}{\sin^2 t} = -\operatorname{ctg} t \end{array} \right] =$

$= -\left(-t \cdot \operatorname{ctg} t + \int \operatorname{ctg} t \, dt \right) =$ ~~arccos cos~~

$= \arccos x \cdot \operatorname{ctg}(\arccos x) - \ln |\sin(\arccos x)|$

(36) $\int \cos(\ln x)\,dx = \left[\begin{array}{l} U=\cos(\ln x) \Rightarrow dU = -\sin(\ln x)\cdot \frac{1}{x}dx \\ dV = dx \Rightarrow V = \int dx = x \end{array} \right] =$

$= x\cos(\ln x) + \int \sin(\ln x)\,\frac{1}{x}\,x\,dx = \left[\begin{array}{l} U=\sin(\ln x) \Rightarrow \\ \Rightarrow dU = \cos(\ln x)\frac{1}{x}dx \\ dV = \int dx = x \end{array} \right] =$

$I = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

$2I = x\cos(\ln x) + x\sin(\ln x)$

$I = \frac{1}{2}\left(x\left(\cos(\ln x) + \sin(\ln x)\right)\right) + C$

Racionalne funkcije

$R(x) = \frac{P_n(x)}{P_m(x)} \rightarrow$ racionalna fja

1) Ako je $n < m \Rightarrow$ prava rac. fja

2) Ako je $n \geq m \Rightarrow R(x) = S_k(x) + \frac{R_l(x)}{Q_m(x)}$ $\quad \underline{l < m}$

4 tipa prostih racionalnih funkcija

(1) $\frac{A}{x-a}$ (2) $\frac{A}{(x-a)^k}$, $k \geq 2$ (3) $\frac{Mx+N}{x^2+px+q}$ (4) $\frac{Mx+N}{(x^2+px+q)^k}$ $k \geq 2$

$k \in \mathbb{N}$, $A, M, N, a, p, q \in \mathbb{R}$

~~(37) $\int \frac{Mx+N}{(x-a)^k} = A$~~

(37) $\int \frac{Mx+N}{x^2+px+q}\,dx = \left[\begin{array}{l} x^2+px+q = \left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right) \\ t = x + \frac{p}{2}\ ;\ dx = dt \\ p^2 - 4q < 0 \Rightarrow 4q - p^2 > 0 \end{array} \right.$ $\quad q - \frac{p^2}{4} = a^2 > 0$

$= \int \frac{M\left(x+\frac{p}{2}\right) + \left(N - \frac{Mp}{2}\right)}{\left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)}\,dx = M\int \frac{t\,dt}{t^2+a^2} + \frac{2N - Mp}{2}\int \frac{dx}{t^2+a^2}$

(38.) $$\int \frac{Mx+N}{(x^2+px+q)^k}\,dx = \int \frac{M\left(x+\frac{p}{2}\right)+N-\frac{Mp}{2}}{\left(\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right)^k}\,dx =$$

$$= M\int \frac{t\,dt}{(t^2+a^2)^k} + \frac{2N-Mp}{2}\int \frac{dt}{(t^2+a^2)^k} \rightarrow I_k$$

$$I_k = \int \frac{dt}{(t^2+a^2)^k} = \left[\begin{array}{l} U=\frac{1}{(t^2+a^2)^k}\ ;\ dU=\frac{-2kt\,dt}{(t^2+a^2)^{k+1}} \\ dV=dt \Rightarrow V=t \end{array}\right] =$$

$$= \frac{t}{(t^2+a^2)^k} + 2k\int \frac{t^2+a^2-a^2}{(t^2+a^2)^k}\,dt = \frac{t}{(t^2+a^2)^k} +$$

$$+ 2k\int \frac{dt}{(t^2+a^2)^k} - 2ka^2\int \frac{dt}{(t^2+a^2)^{k+1}}$$

$$I_k = \frac{t}{(t^2+a^2)^k} + 2kI_k - 2ka^2 I_{k+1}$$

$$I_{k+1} = \frac{1}{2ka^2}\cdot\frac{t}{(t^2+a^2)^k} + \frac{2k-1}{2ka^2} I_k, \quad k=1,2\ldots$$

$$I_1 = \int \frac{dt}{t^2+a^2} = \frac{1}{a}\,\mathrm{arctg}\,\frac{t}{a}+C$$

$$I_2 = \frac{1}{2ka^2}\cdot\frac{t}{t^2+a^2} + \frac{1}{2a^2}\cdot\frac{1}{a}\,\mathrm{arctg}\,\frac{t}{a}+C$$

...

Rekurentna formula :) I_3

I_4 . . .

(39) $\int \frac{x^4-3x}{x^3+1}\,dx = \left[\begin{array}{l} (x^4-3x) : (x^3+1) = x \\ \underline{-x^4 - x} \\ \quad / \quad -4x \end{array} \right] = \int \frac{(x^3+1)x - 4x}{x^3+1}\,dx =$

$= \int x\,dx - 4\int \frac{x}{x^3+1}\,dx = \frac{x^2}{2} - 4 \int \frac{x}{\underbrace{(x+1)(x^2-x+1)}}\,dx =$

$P = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ $\longrightarrow$ onoliko koeficijenata koliko višestrukosti nule tog polinoma

$P = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad / \cdot (x+1)(x^2-x+1)$

$-4x = A(x^2-x+1) + (Bx+c)(x+1)$

$-4x = (A+B)x^2 + (-A+B+C)\,x + A+C$

$A+B=0 \Rightarrow B=-A$

$-A+B+C=-4$

$A+C=0 \Rightarrow C=-A$

$-3A=-4$

$A=\frac{4}{3}$

$B=-\frac{4}{3} \qquad C=-\frac{4}{3}$

$= \frac{x^2}{2} + \cancel{4}\int \frac{\frac{4}{3}\,dx}{x+1} - \frac{4}{3}\int \frac{x+1}{x^2-x+1}\,dx =$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{4}{6}\int \frac{2x-1+3}{x^2+x+1}\,dx =$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{4}{6}\int \frac{(2x-1)\,dx}{x^2-x+1} - 2\int \frac{dx}{x^2-x+1} =$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{4}{6}\ln|x^2-x+1| - 2\int \frac{dx}{x^2-x+1} =$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{4}{6}\ln|x^2-x+1| - 2\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} =$

$= \frac{x^2}{2} + \frac{4}{3}\ln|x+1| - \frac{2}{3}\ln|x^2-x+1| - \frac{4}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2x-1}{\sqrt{3}} + C$

$$\# \quad \frac{-4x}{(x+1)^2(x^2+x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}$$

$$\# \quad \frac{1}{(x+1)^5} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+1)^4} + \frac{E}{(x+1)^5}$$

$D=-3,\ D<0 \Rightarrow$ nema nula

40. $$\int \frac{1-3x}{x^2+x+1}\,dx = \left[\begin{array}{l} x^2+x+1=t \\ (2x+1)dx=dt \\ -\frac{3}{2}(2x+1)+1+\frac{3}{2} \end{array}\right] =$$

$$= \int \frac{\frac{5}{2}-\frac{3}{2}(2x+1)}{x^2+x+1}\,dx = \frac{5}{2}\int \frac{dx}{x^2+x+1} - \frac{3}{2}\int \frac{2x+1}{x^2+x+1}\,dx =$$

$$= \frac{5}{2}\int \frac{dx}{(x+\frac{1}{2})^2+1-\frac{1}{4}} - \frac{3}{2}\int \frac{dt}{t} = \frac{5}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} - \frac{3}{2}\ln|x^2+x+1|$$

$$= \frac{5}{2}\cdot\frac{4}{3}\int \frac{dx}{\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)^2+1} - \frac{3}{2}\ln|x^2+x+1| =$$

$$= \frac{10}{3}\int \frac{dx}{\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)^2+1} - \frac{3}{2}\ln|x^2+x+1| = \left[\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}=t\right] =$$

$$= \frac{10}{3}\cdot\frac{1}{\frac{2}{\sqrt{3}}}\,\mathrm{arctg}\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right) - \frac{3}{2}\ln|x^2+x+1| + C$$

$D=0$

41. $$\int \frac{3x-2}{(x+1)^2}\,dx = \left[\begin{array}{ll} x+1=t & dx=dt \\ x=t-1 & \end{array}\right] =$$

$$= \int \frac{3t-3-2}{t^2}\,dt = \int \frac{3t-5}{t^2}\,dt = 3\int\frac{dt}{t} - 5\int\frac{dt}{t^2} =$$

$$= 3\ln|x+1| + 5\cdot\frac{1}{x+1} + C$$

$$D = 1+8 = 9 \Rightarrow \text{ima nule} \quad x_{1,2} = \frac{+1 \pm 3}{2} = \begin{cases} -1 \\ 2 \end{cases}$$

42. $\cancel{\int \frac{2x+1}{(x+1)^2}}$ $I = \int \frac{2x+1}{x^2-x-2}\,dx = \int \frac{2x+1}{(x-2)(x+1)}\,dx =$

metoda neodređenih koeficijenata

$$\frac{2x+1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \quad / \cdot (x-2)(x+1)$$

$$2x+1 = A(x+1) + B(x-2)$$

$$2x+1 = Ax + Bx + A - 2B$$

$$A + B = 2 \Rightarrow A = 2 - B$$

$$A - 2B = 1$$

$$2 - B - 2B = 1$$

$$-3B = -1 \Rightarrow B = \frac{1}{3} \quad A = \frac{5}{3}$$

$$= \frac{5}{3}\int \frac{dx}{x-2} + \frac{1}{3}\int \frac{dx}{x+1} = \frac{5}{3}\ln|x-2| + \frac{1}{3}\ln|x+1| + C$$

43. $\int \frac{x}{(x-1)(x+1)^2}\,dx =$

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \quad / \cdot (x-1)(x+1)^2$$

$$x = A(x+1)(x+1) + B(x-1)(x+1) + C(x-1)$$

$$x = A(x^2+2x+1) + B(x^2-1) + C(x-1)$$

$$x = (A+B)x^2 + (2A+C)x + A - B - C$$

$$A + B = 0$$

$$2A + C = 1$$

$$A - B - C = 0$$

$$A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = \frac{1}{2}$$

$$= \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{(x+1)^2} =$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2}\cdot\left(-\frac{1}{x+1}\right) + C$$

(44) $\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)} =$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \quad / (x+1)(x^2-x+1)$$

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$1 = x^2(A+B) + (-A+B+C)x + A + C$$

$A+B=0 \Rightarrow B=-A$

$-A+B+C=0$

$A+C=1$

$-2A+C=0$

$A+C=1 \Rightarrow C = 1-A$

$$-2A+1-A=0 \rightarrow -3A=-1 \Rightarrow A=\frac{1}{3}$$

$$B=-\frac{1}{3} \quad C=\frac{2}{3}$$

$$= \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{x-2}{x^2-x+1}\,dx = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{x-2}{x^2-x+1}\,dx =$$

$$= \boxed{\begin{matrix} x^2-x+1=t \\ (2x-1)\,dx = dt \end{matrix}} = \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\cancel{\frac{(2x-1)-x-1}{x^2-x+1}}\,dx =$$

$-\frac{1}{2}-2=$

$$= \cancel{\frac{1}{3}\ln|x+1| - \frac{1}{3}\cdot\int\frac{1}{2}(2x-1)} - \quad =$$

$-\frac{1}{2}-\frac{4}{4}=$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int\frac{\frac{2x-4}{2}}{x^2-x+1}\,dx = \frac{1}{3}\ln|x+1| - \frac{1}{6}\int\frac{2x-1-3}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\int\frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int\frac{dx}{x^2-x+1}\,dx =$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} =$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{1}{2}\int\frac{dx}{\frac{3}{4}\left(\frac{2x}{\sqrt{3}}-\frac{2}{\sqrt{3}}\right)^2+\frac{3}{4}} =$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{\cancel{4}^{2}}{3}\cdot\frac{1}{\cancel{2}_{1}}\cdot\int\frac{dx}{\left(\frac{2x}{\sqrt{3}}-\frac{\cancel{2}}{\sqrt{3}}\right)^2+1} = \boxed{\begin{matrix} z=\frac{2x}{\sqrt{3}}-\frac{2}{\sqrt{3}} \\ dx=\frac{\sqrt{3}}{2}dz \end{matrix}}$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln|x^2-x+1| + \frac{\cancel{2}}{3}\cdot\frac{\sqrt{3}}{\cancel{2}}\cdot\operatorname{arctg}\left(\frac{2}{\sqrt{3}}x-\frac{\cancel{2}}{\sqrt{3}}\right)+C$$

$D = 1-4=3 < 0$

(45) $\int \frac{x-2}{(x^2-x+1)^2}\,dx$

ne može ovako

$\cancel{\frac{x-2}{(x^2-x+1)^2} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{(x^2-x+1)^2}}$

$$\int \frac{x-2}{(x^2-x+1)^2}\,dx = \left[\begin{array}{l} x^2-x+1=t \\ (2x-1)\,dx = dt \end{array}\right] = \int \frac{\frac{2x-4}{2}}{(x^2-x+1)^2}\,dx =$$

$$= \frac{1}{2}\int \frac{2x-1-3}{(x^2-x+1)^2}\,dx = \frac{1}{2}\int \frac{2x-1}{(x^2-x+1)^2}\,dx - \frac{3}{2}\int \frac{dx}{(x^2-x+1)^2} =$$

$$= \left[\begin{array}{l} x^2-x+1=t \\ (2x-1)\,dx = dt \end{array}\right] = \frac{1}{2}\int \frac{dt}{t^2} - \frac{3}{2}\int \frac{dx}{\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)^2} =$$

$$= -\frac{1}{2}\cdot\frac{1}{x^2-x+1} - \frac{\cancel{3}}{\cancel{2}}\,\frac{\cancel{4}^{\,2}}{\sqrt{3}}\int \frac{dx}{\left(\left(\frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)^2+1\right)^2} =$$

$$= -\frac{1}{2}\,\frac{1}{x^2-x+1} - \sqrt{3}\int \frac{dx}{\left(\left(\frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)^2+1\right)^2} = \left[\begin{array}{l} \frac{2x}{\sqrt{3}}-\frac{1}{\sqrt{3}} = z \\ dx = \frac{\sqrt{3}}{2}\,dz \end{array}\right] =$$

$$= -\frac{1}{2}\,\frac{1}{x^2-x+1} - \cancel{2}\sqrt{3}\,\frac{\sqrt{3}}{\cancel{2}}\int \frac{dx}{(z^2+1)^2} =$$

$$= -\frac{1}{2}\,\frac{1}{x^2-x+1} - 3\int \frac{dx}{(z^2+1)^2} \rightarrow \text{može } \frac{1+z^2-z^2}{(z^2+1)^2} =$$

$$\# \int \frac{dx}{(1+x^2)^3} \rightarrow \text{parc integracija}$$

Rekurentna formula

$$\int \frac{1+x^2-x^2}{(1+x^2)^3}\,dx$$